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A one-dimensional piston problem of gasdynamics

By R. McLAUGHLIN

Department of Mathematics, Cranfield Institute of Technology,
Cranfield, Bedford MK43 0AL, England†

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This paper considers the case of a one-dimensional piston moving outwards with a speed proportional to r^α and driving a strong shock into a non-uniform ambient gas whose density is initially proportional to $r^{-k}, k > 0$. This problem is connected with that studied by Grundy & McLaughlin (1977), who effectively discussed the case $\alpha = 0$. We discover further important uses of the Sedov similarity solutions and find k_c, the upper limit to k for the shock path to be asymptotically similar to the piston path.

1. Introduction

In a recent paper (Grundy & McLaughlin 1977), the authors investigated the unsteady expansion of a uniform source gas into a non-uniform ambient atmosphere, a problem which is equivalent to that of a one-dimensional piston moving outwards with constant speed into a non-uniform ambient gas. Assuming an asymptotically constant shock velocity, these authors obtained the large time solution by the method of matched expansions and found an upper limit to k for a successful match. For larger k the assumption on the shock velocity was reviewed. An expanded version of this work was given by McLaughlin (1975), who indicated how to study the problem of a one-dimensional piston moving outwards with speed $A(r'/L)^\alpha$ into an ambient gas of initial density $\rho_0^*(r'/L)^{-k}$, where r' is the dimensional spatial co-ordinate, L is the initial piston radius, ρ_0^* is the initial density at $r' = L$ and $A > 0$, $\alpha > 0$ and $k > 0$ are constants. This is the problem that we discuss here and it is our aim to investigate the large time solution and thus establish k_c, the upper limit on k for the asymptotic shock velocity to be of the form

$$V'(r'/L) = Ab_0(r'/L)^\alpha + \dots,$$

and also to evaluate b_0.

We omit the matching details as they are essentially the same as those in Grundy & McLaughlin (1977). The zeroth-order inner solution (valid near the shock) is examined using the similarity solutions of Sedov (1959, p. 146) and it soon becomes clear that there is an upper limit k_c to k for a similarity solution to exist. As an illustration, we calculate k_c as a function of σ and b_0 as a function of k for various values of α.

† Present address: Department of Mathematics and Computing, Wigan College of Technology, Wigan, Greater Manchester, England.

2. Equations, boundary conditions and similarity solution

The dimensional quantities, the primed variables, are related to the non-dimensional quantities, the unprimed variables, by

$$u' = Au, \quad \rho' = \rho_0^* \rho, \quad p' = \rho_0^* A^2 p, \quad r' = rL, \quad t' = Lt/A, \quad V' = AV,$$

where u, ρ and p are respectively the gas velocity, density and pressure and r, t and V are the radial co-ordinate, time and the shock velocity.

The equations governing the motion of the gas are

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\rho r^\sigma) + \frac{\partial}{\partial r}(\rho u r^\sigma) &= 0, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0, \\ \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)(p\rho^{-\gamma}) &= 0, \end{aligned}\right\} \qquad (2.1)$$

where γ is the constant ratio of specific heats of the gas and σ, the geometry index, takes the values 0, 1 and 2 respectively for plane, cylindrical and spherical symmetry.

The boundary condition on the piston is

$$u = r^\alpha \quad \text{on} \quad dr/dt = r^\alpha,$$

and on letting $a_0'/V' \to 0$, where a_0' is the sound speed of the undisturbed gas, the Rankine–Hugoniot shock relations become

$$\left.\begin{aligned} u &= 2V/(\gamma+1), \\ \rho &= r^{-k}(\gamma+1)/(\gamma-1), \\ p &= 2V^2 r^{-k}/(\gamma+1), \end{aligned}\right\} \qquad (2.2)$$

which apply on $dr/dt = V$.

We now assume that the shock path is asymptotically similar to the piston path, i.e. we let

$$V(r) = b_0 r^\alpha \{1 + b_1 r^{\beta_1} + \ldots\}, \quad \mathrm{Re}\,\beta_1 < 0, \qquad (2.3)$$

and we can construct asymptotic expansions of the solution to the boundary-value problem. There are, of course, difficulties which arise in the matching but these are similar to those in Grundy & McLaughlin (1977) and in the hypersonic small disturbance theory of Freeman (1965), Ellinwood (1967) and Stewartson & Thompson (1968, 1970). All we wish to say about the matching is that the results of Grundy & McLaughlin (1977) for $k < \sigma + 1$ can be recovered immediately from our analysis by setting $\alpha = 0$ but for $k > \sigma + 1$ no immediate recovery can be made. Also, as in Grundy & McLaughlin (1977), matching the zeroth-order inner terms with the outer expansion (valid near the piston) gives the constant b_0, matching to first order produces an eigenvalue problem for β_1 whilst b_1 cannot be determined by the asymptotic analysis alone.

Before we attempt to calculate b_0, however, we must establish the existence of a solution to the zeroth-order inner problem. We observe that this solution is, in a different description, the similarity or progressing-wave solution of one-dimensional gasdynamics, e.g. see Courant & Friedrichs (1948, p. 419) or Sedov (1959, p. 146).

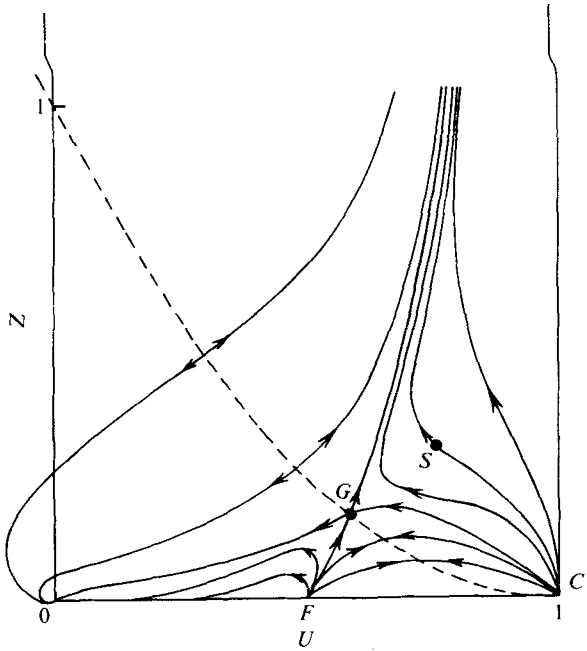

FIGURE 1. Integral curves in the phase plane of Z and U. Arrows indicate the direction of increasing λ.

Following Sedov we let

$$u = \delta(r/t)\,U(\lambda), \quad a^2 = \delta^2(r/t)^2\,Z(\lambda), \tag{2.4}$$

where $a^2 = \gamma p/\rho$ and $\lambda = rt^{-\delta}$ is the similarity variable with $\delta = 1/(1-\alpha) > 1$.

Equations (2.1) together with (2.4) eventually produce a single first-order differential equation in Z and U:

$$\frac{dZ}{dU} = \frac{ZS(U,Z)}{(1-U)\,Q(U,Z)}, \tag{2.5}$$

with

$$S = \{2(U-\delta^{-1}) + (\gamma-1)(\sigma+1)\,U\}(1-U)^2 + (\gamma-1)\,U(U-\delta^{-1})(1-U) - Z\{2(U-\delta^{-1}) + K(\gamma-1)/\delta\},$$

$$Q = U(U-\delta^{-1})(1-U) + Z\{(\sigma+1)\,U - K/\delta\},$$

$$K = \{2+\delta(k-2)\}/\gamma.$$

The strong shock is located in the phase plane of Z and U at S, where from (2.2) and (2.4)

$$Z = Z_S = 2\gamma(\gamma-1)/(\gamma+1)^2, \quad U = U_S = 2/(\gamma+1),$$

and the piston at C, where

$$Z = Z_C = 0, \quad U = U_C = 1.$$

Figure 1 shows a typical phase-plane diagram. In his thesis (McLaughlin 1975), the present writer discusses the case $k > \sigma+1$, $\delta > 1$ in great detail and the only difference for $k < \sigma+1$ is that there is a family of integral curves leaving the node C perpendicular to the U axis. It is important to note here that G is a saddle point.

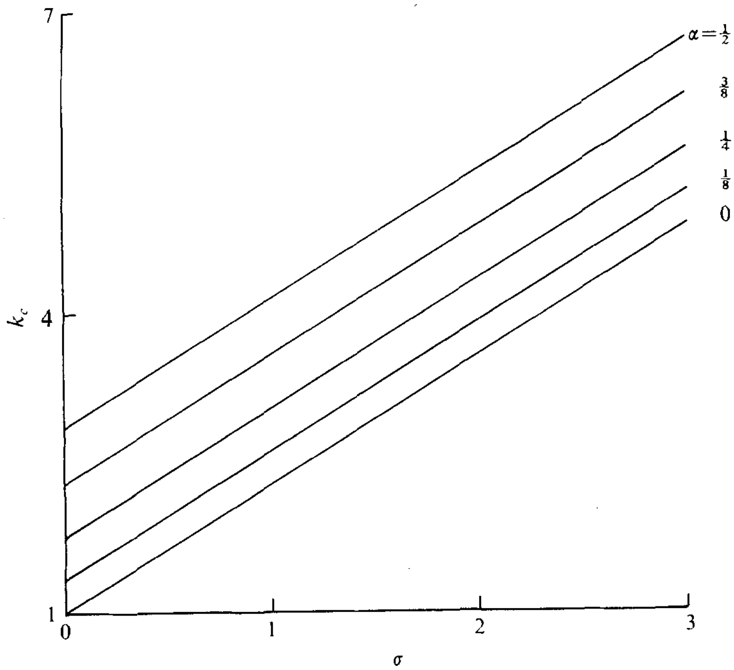

FIGURE 2. The variation of k_c with σ for $\gamma = \frac{5}{3}$ and various values of α.

The argument now used to establish k_c is entirely similar to that used by Grundy & McLaughlin (1977). We can see that, for sufficiently small k, there is an integral curve of (2.5) joining C to S and along this curve λ varies monotonically since the integral curve Δ joining the singular points F, G and D lies to the left of S. As k increases, Δ moves to the right until, at $k = k_c$, it passes through S. Clearly for $k > k_c$ no solution curve exists and thus k_c is the upper limit for the assumption (2.3) to be valid. The function k_c must be evaluated numerically, e.g. see Grundy & McLaughlin (1977) or McLaughlin (1975), unless $\alpha = \alpha^* = (\gamma-1)/(\gamma+1)$. In this special case Δ is the line $U = 2/(\gamma+1)$ and hence

$$k_c = k^* = 2\{\gamma(\sigma+1)+(\gamma-1)\}/(\gamma+1).$$

The variation of k_c with σ for various values of α for $\gamma = \frac{5}{3}$ is shown in figure 2.

3. Zeroth-order inner solution and calculation of b_0

Having verified that a solution to the zeroth-order problem exists for $k < k_c$, we can calculate b_0. The coefficient b_0 can, in theory, be obtained from the similarity solution but, as there is a singularity at C in (2.5), it is far easier in practice to obtain it using the particle-path co-ordinate.

Following Grundy & McLaughlin (1977) we introduce, at the expense of time t, ψ and then ϕ, where

$$\partial\psi/\partial r = \rho r^\sigma, \quad \partial\psi/\partial t = -\rho u r^\sigma,$$

$$\phi = \begin{cases}\{1+(\sigma+1-k)\psi\}\, r^{k-\sigma-1} & \text{for} \quad k \neq \sigma+1, \\ e^{\psi}/r \quad \text{for} \quad k = \sigma+1. \end{cases}$$

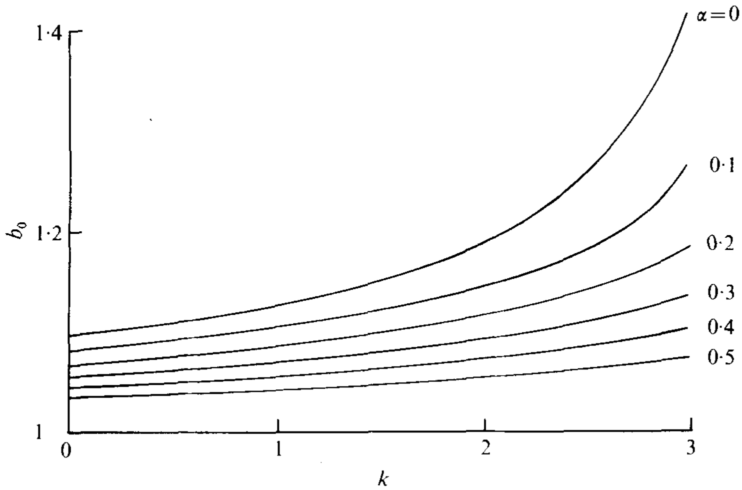

FIGURE 3. The variation of b_0 with k for $\sigma = 2$, $\gamma = \frac{5}{3}$ and various values of α.

The shock then lies on $\phi = 1$ and for $r \to \infty$ the piston lies on

$$\phi = \phi_0 = \begin{cases} 0 & \text{for} \quad k \leqslant \sigma+1, \\ \infty & \text{for} \quad k > \sigma+1. \end{cases}$$

For the zeroth-order inner solution only we substitute

$$u = \frac{2}{\gamma+1} b_0 r^{\alpha} U_0(\phi), \quad p = \frac{2}{\gamma+1} b_0^2 r^{2\alpha-k} P_0(\phi)$$

and
$$\rho = \frac{\gamma+1}{\gamma-1} r^{-k} R_0(\phi)$$

into (2.1) to obtain, for $k \neq \sigma+1$,

$$\left.\begin{aligned} &(\sigma+\alpha-k) R_0 U_0 - (\sigma+1-k)\phi(R_0 U_0)' + (\sigma+1-k)\left(\frac{\gamma+1}{\gamma-1}\right) R_0^2 U_0' = 0, \\ &(\sigma+1-k) R_0 U_0 \phi U_0' - \alpha R_0 U_0^2 + \tfrac{1}{2}(k-2\alpha)(\gamma-1) P_0 \\ &\qquad + \tfrac{1}{2}(\gamma-1)(\sigma+1-k)\phi P_0' - \tfrac{1}{2}(\gamma+1)(\sigma+1-k) R_0 P_0' = 0, \\ &P_0 = R_0^{\gamma} \phi^{[k(\gamma-1)+2\alpha]/(\sigma+1-k)}, \end{aligned}\right\} \qquad (3.1)$$

with equivalent equations when $k = \sigma+1$.

The boundary conditions at the shock are

$$U_0(1) = P_0(1) = R_0(1) = 1 \qquad (3.2)$$

and matching requires that

$$2(\gamma+1)^{-1} b_0 U_0 \to 1 \quad \text{as} \quad \phi \to \phi_0.$$

Obviously b_0 is obtained by integrating (3.1) numerically from $\phi = 1$, using (3.2), to $\phi = \phi_0$ with the result

$$b_0 = (\gamma+1)/2U_0(\phi_0).$$

We illustrate this by taking the case $\sigma = 2$, $\gamma = \frac{5}{3}$; the graphs of b_0 *vs.* k for various values of α being shown in figure 3.

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